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The Covariant Derivative in the Affine Approach to General Relativity

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Abstract

The affine presentation of general relativity is considered and a possible generalisation of the definition of covariant derivative is proposed. Under certain weak symmetry conditions it is shown that the only theories resulting from this generalisation are general relativity and Weyl's theory, of which general relativity arises in the most natural way.

1. *Introduction*

The affine geometrical approach to general relativity requires that covariant differentiation be defined in terms of the fundamental symmetric tensor g_{ij}. Once this is done, the affine paths become the trajectories of the theory and the field equations are constructed from tensors depending only upon the concept of parallel displacement. General relativity therefore follows from purely affine properties once the expressions for covariant derivatives in terms of Christoffel brackets are obtained. These expressions are often derived from a requirement of symmetry in the affine connections, and from the four axioms for covariant differentiation given in Section 2 (Schouten, 1954).

Other physical theories have been obtained by dropping the requirements of symmetry in g_{ij} and in the affine connection, as in Einstein's and in Schrödinger's unified theories (Schrödinger, 1950). Any other affine theories which are based on a symmetric fundamental tensor g_{ij}, will necessarily derive from other expressions for covariant derivative. Thus, by weakening the axioms for covariant differentiation (with the exception of the affine condition, axiom 1), affine theories of a more general nature are obtained. In this way, Weyl's theory is such a generalisation, in which covariant differentiation does not satisfy axiom 4 (Schouten, 1954; Weyl, 1952). In this paper the generalisations resulting from a weakening of axiom 3, the product rule, are investigated. With only a very weak assumption of symmetry, a set of tensor connections are obtained, expressing the covariant derivatives of tensors of each order in terms of g_{ij} and a single unspecified vector field a_l. Of the requirements of symmetry which may be imposed on the connections that are developed, the most natural one leads to a theory physically equivalent to general relativity, and the others produce the connection on which Weyl's theory is based.

2. *The Axioms for Covariant Differentiation*

The expressions for covariant differentiation in terms of Christoffel brackets may be derived from a requirement of symmetry in the affine connection, together with the following axioms (Schouten, 1954):

(1) The covariant derivative of a tensor $A^P{}_Q$ is produced from the ordinary derivative together with terms linear in $A^P{}_Q$, the result being a tensor. (The notation adopted here is that in which capital letters represent ordered sets of suffixes.)

$$A^P{}_{Q;l} = A^P{}_{Q,l} + \Gamma^{P\cdot}{}_{Q.I.}{}^{J\cdot}{}_l A^I{}_J \tag{2.1}$$

Here the $\Gamma^{P\cdot}{}_{Q.I.}{}^{J\cdot}{}_l$ stand for a whole array of affine connections, which so far are not related. Even for vectors, the affine connections involved in the definitions of $A^i{}_{;l}$ and $A_{i;l}$ are not assumed to be related.

(2) The covariant derivative of a scalar is the ordinary derivative:

$$\phi_{;l} = \phi_{,l} \tag{2.2}$$

(3) The covariant derivative of a product is given by Leibnitz' rule, in terms of the covariant derivatives of each factor:

$$(A^P{}_Q B^R{}_S)_{;l} = A^P{}_{Q;l} B^R{}_S + A^P{}_Q B^R{}_{S;l} \tag{2.3}$$

(4) Raising and lowering of tensor suffixes by means of g_{ij} is an operation which commutes with covariant differentiation, e.g.:

$$g_{RP} A^P{}_{Q;l} = A_{RQ;l} \tag{2.4}$$

Here, g_{RP} denotes the product $g_{i_1 j_1} g_{i_2 j_2} \ldots g_{i_n j_n}$ where R is the ordered set $(i_1, i_2, \ldots i_n)$ and P is the ordered set $(j_1, j_2, \ldots j_n)$.

In this paper, Leibnitz' rule is not assumed. However, equation (2.1) of axiom 1, allows us to write for the derivative of the product of tensors $A^P{}_Q$ and $B^R{}_S$:

$$(A^P{}_Q B^R{}_S)_{;l} = A^P{}_{Q;l} B^R{}_S + A^P{}_Q B^R{}_{S;l} + U^P{}_Q{}^{/R}{}_{S.I.}{}^{J/}{}_K{}^{L\cdot}{}_l A^I{}_J B^K{}_L, \tag{2.5}$$

where $U^P{}_Q{}^{/R}{}_{S.I.}{}^{J/}{}_K{}^{L\cdot}{}_l$ is a set of functions of position, not depending on $A^P{}_Q$ or $B^R{}_S$. Equation (2.5) represents the generalised product rule, weakened in the only possible way consistent with the other axioms.

Wherever possible, only products of contravariant tensors will be considered, so that $U^P{}_\phi{}^{/R}{}_{\phi.I}{}^{\phi/}{}_K{}^{\phi\cdot}{}_l$ (where ϕ is the null set) is abbreviated to $U^{P/R\cdot}{}_{I/K.l}$ i.e.

$$(A^P B^R)_{;l} = A^P{}_{;l} B^R + A^P B^R{}_{;l} + U^{P/R\cdot}{}_{I/K.l} A^I B^K$$

3. *Differentiation Without the Product Rule*

We first note that, according to equation (2.5) and axiom 4, the U-tensor depends only upon the ranks of the factors involved, and not upon their tensor characters. Suffixes may therefore be raised and lowered in U by means of g_{ij}.

From (2.5), the U-tensors may be expressed in terms of connections through the relation

$$\Gamma^{(PR)\cdot}{}_{(QS).(IK).}{}^{(JL)\cdot}{}_{l} = \delta^R{}_K \delta^L{}_S \Gamma^{P\cdot}{}_{Q.I.}{}^{J\cdot}{}_{l} + \delta^P{}_I \delta^J{}_Q \Gamma^{R\cdot}{}_{S.K.}{}^{L\cdot}{}_{l} + \\ + U^P{}_Q{}^{/R}{}_{S.I.}{}^{J/}{}_K{}^{L\cdot}{}_l \tag{3.1}$$

Here, (PR) represents the ordered set of suffixes P, followed by the ordered set R, etc. The condition that the correct covariant derivative for a lower rank tensor must be obtained when on contraction of suffixes in the covariant derivative of a tensor, gives the equation

$$\Gamma^{(PR)\cdot}{}_{(QR).(IK).}{}^{(JL)\cdot}{}_{l} = \delta^L{}_K \Gamma^{P\cdot}{}_{Q.I.}{}^{J\cdot}{}_{l} \tag{3.2}$$

Equation (3.2) allows us to express axiom 2 by the single relation:

$$\Gamma^{p\cdot}{}_{p.i.}{}^{j\cdot}{}_{l} = 0 \tag{3.3}$$

and using this equation, we deduce that

$$U^P{}_Q{}^{/R}{}_{R.I}{}^{J/}{}_K{}^{L\cdot}{}_l = 0 \tag{3.4}$$

by putting $R = S$ in (3.1). Equations (3.1) and (3.2) may now be used to derive the relation

$$U^{P/QR}{}_{R.I/JK}{}^{L\cdot}{}_l = \delta^L{}_K\, U^{P/Q\cdot}{}_{I/J.l} \tag{3.5}$$

so that $U^{P/\phi\cdot}{}_{I/\phi.l} = 0$, i.e. Leibnitz' product rule applies to products of scalars with tensors. In fact, since the whole argument may be reversed, the relation (3.4) is equivalent to axiom 2.

Now equation (3.1) is used to reduce the connections $\Gamma^{PQR\cdot}{}_{\phi.IJK.}{}^{\phi\cdot}{}_l$ to three connections of the type $\Gamma^{P\cdot}{}_{\phi.I.}{}^{\phi\cdot}{}_l$. By making the reduction in two ways, we obtain the relation

$$U^{P/QR\cdot}{}_{I/JK.l} + \delta^P{}_I\, U^{Q/R\cdot}{}_{J/K.l} = U^{Q/RP\cdot}{}_{J/KI.l} + \delta^Q{}_J\, U^{R/P\cdot}{}_{K/I.l} \tag{3.6}$$

Apart from the relations already given, there remain only the obvious symmetry relations

$$U^{P/Q\cdot}{}_{I/J.l} = U^{Q/P\cdot}{}_{J/I.l} \tag{3.7}$$

$$U^{P/QR\cdot}{}_{I/JK.l} = U^{P/RQ\cdot}{}_{I/KJ.l} \tag{3.8}$$

By making use of the relations (3.6) to (3.8), all U-tensors of a certain tensor order are expressible in terms of any particular one of that order, together with U-tensors of lower orders. Apart from the restriction (3.4), one tensor of each order may thus be chosen quite freely. In Section 4, we shall consider U-tensors which satisfy the one extra symmetry relation

$$U^{P/QR.}{}_{I/JK.l} = U^{Q/PR.}{}_{J/IK.l} \tag{3.9}$$

where, of course, P and Q have the same number of members. In this case, we show that all U-tensors are determined in terms of $U^{p/q.}{}_{i/j.l}$.

4. *The Symmetric U-tensor*

By using the extra condition (3.9), it may now be shown that the U-tensors are equal to vectors multiplied by delta symbols, and further, that the vectors are equal to an unspecified vector, a_l, or to zero. By substituting (3.9) into (3.6), we have

$$\delta^P{}_I \, U^{Q/R.}{}_{J/K.l} = \delta^Q{}_J \, U^{R/P.}{}_{K/I.l} \tag{4.1}$$

whence, putting P and I equal to M,

$$U^{Q/R.}{}_{J/K.l} = \delta^Q{}_J \, U^{R/M.}{}_{K/M.l} = \delta^Q{}_J \, \delta^R{}_{Kl}$$

Using the symmetry relation (3.7),

$$\delta^Q{}_J \, S^R{}_{Kl} = \delta^R{}_K \, S^Q{}_{Jl}$$

and putting Q and J equal to N,

$$S^R{}_{Kl} = \delta^R{}_K \, S^N{}_{Nl} = \delta^R{}_K \, a_l$$

$U^{P/Q.}{}_{I/J.l}$ may therefore be written in terms of a vector $a_l{}^{(p,q)}$, depending upon p & q, the number of members of P, Q respectively:

$$U^{P/Q.}{}_{I/J.l} = \delta^P{}_I \, \delta^Q{}_J \, a_l{}^{(p,q)} \tag{4.2}$$

Equations (3.4) and (3.5) give

$$a_l{}^{(p,0)} = 0, \tag{4.3}$$

$$a_l{}^{(p,q)} = a_l{}^{(p+2,q)} \tag{4.4}$$

From these two relations, together with the symmetry relation, $a_l{}^{(p,q)} = a_l{}^{(q,p)}$, we deduce that $a_l{}^{(p,q)} = 0$ unless both p and q are odd, when $a_l{}^{(p,q)} = a_l{}^{(1,1)}$ ($=a_l$). The product rules are now completely determined in terms of the single vector a_l.

The general connection, $\Gamma^{P.}{}_{Q.I.}{}^{J.}{}_l$ can now be conveniently expressed in terms of the two basic connections involved in the derivatives of contravariant and covariant vectors. For simplicity we write

$$\Gamma^{p.}{}_{\phi.i.}{}^{\phi.}{}_l = \Gamma^p{}_{il}$$
$$\Gamma^{\phi.}{}_{q.\phi.}{}^{j.}{}_l = -\Omega^j{}_{ql}$$

Using (3.1), the general connection may be expressed:

$$\Gamma^{P.}{}_{Q.I.}{}^{J.}{}_l = \delta^P{}_I \left(\sum_{\substack{p_r \epsilon P \\ i_r \epsilon I}} \Gamma^{p_r}{}_{i_r l} / \delta^{p}{}_{r i_r l} - \sum_{\substack{q_r \epsilon Q \\ j_r \epsilon J}} \Omega^{j_r}{}_{q_r l} / \delta^{j_r}_{q_r} + m a_l \right) \tag{4.5}$$

where m is $[(p+q)/2]$. In fact, this general connection may be expressed in terms of a_l and Γ alone, since by (3.3), we have the relation

$$\Omega^p{}_{il} - \Gamma^p{}_{il} = \delta^p{}_i a_l \tag{4.6}$$

An expression for the covariant derivative of the tensor g_{PQ}, from which will be determined expressions for the connection (4.5) in terms of g_{ij} will be determined in the following section, may be derived by considering axiom 4 in relation to (2.5). In general,

$$(A^I g_{IQ})_{;l} = A^I{}_{;l} g_{IQ} + A^I g_{IQ;l} + U^{I/}{}_{IQ.P}{}^{/JK\cdot}{}_l g_{JK} A^P$$

From this equation axiom 4 is satisfied if

$$g_{PQ;l} = -U^{I/}{}_{IQ.P/}{}^J{}_{J.l}$$

or, by means of (3.5) and (3.6)

$$g_{PQ;l} = U_{P/Q.}{}^{I/}{}_{I.l} - U^{I/}{}_{I.P/Q.l}$$

For the U-tensors obtained in this section (4.2), we deduce that

$$g_{PQ;l} = 0 \tag{4.7}$$

5. *The Connections in Terms of g_{ij} and a_l*

Starting from equation (4.7), $g_{PQ;l} = 0$, $\Gamma^i{}_{jk}$ may be obtained in terms of g_{ij} and a_l, the other connections in (2.2) following from (4.5) and (4.6). Equation (4.7) gives, on taking $P = \{p\}$, $Q = \{q\}$,

$$g_{pq,l} - \Gamma^i{}_{pl}\, g_{iq} - \Gamma^i{}_{ql}\, g_{pi} - a_l\, g_{pq} = 0$$

From this expression, we deduce in the usual manner (Schouten, 1954; Schrödinger, 1950) that

$$\Gamma^i{}_{jk} = \left\{ {i \atop j\,k} \right\} + g^{ia} g_{kb} \Gamma^b{}_{\underset{\vee}{ja}} + g^{ia} g_{jb} \Gamma^b{}_{\underset{\vee}{ka}} + \Gamma^i{}_{\underset{\vee}{jk}} - \tfrac{1}{2}(\delta^i{}_j a_k + \delta^i{}_k a_j - g_{jk} a^i) \tag{5.1}$$

where the antisymmetric part $\Gamma^i{}_{\underset{\vee}{jk}}$ remains completely arbitrary.

In the affine approach to general relativity, the affine connection $\Gamma^i{}_{jk}$, derived from axioms 1–4, is restricted to be symmetric in j and k. In the case of the connections given by (4.5), a restriction of this sort will involve the two connections $\Gamma^i{}_{jk}$ and $\Omega^i{}_{jk}$. The most natural restriction is one which maintains symmetry between Γ and Ω:

$$\Gamma^i{}_{\underset{\vee}{jk}} + \Omega^i{}_{\underset{\vee}{jk}} = 0 \tag{5.2}$$

a relation which reduces to $\underset{\vee}{\Gamma}{}^i{}_{jk} = 0$, when $a_l = 0$, since in this case $\underset{\vee}{\Gamma}{}^i{}_{jk} = \underset{\vee}{\Omega}{}^i{}_{jk}$. Using (5.2), and (4.6), we obtain from (5.1) the expressions

$$\Gamma^i{}_{jk} = \left\{ {i \atop j\,k} \right\} - \tfrac{1}{2}\delta^i{}_j a_k \tag{5.3}$$

and

$$\Omega^i{}_{jk} = \left\{ {i \atop j\,k} \right\} + \tfrac{1}{2}\delta^i{}_j a_k$$

The connections (5.3) give the following formulae for the covariant derivatives of tensors of each order:

$$\begin{aligned} \phi_{;m} &= \phi_{/m}, & A^i{}_{;m} &= A^i{}_{/m} - \tfrac{1}{2}a_m A^i \\ A^{ij}{}_{;m} &= A^{ij}{}_{/m}, & A^{ijk}{}_{;m} &= A^{ijk}{}_{/m} - \tfrac{1}{2}a_m A^{ijk} \\ A^{ijkl}{}_{;m} &= A^{ijkl}{}_{/m}, \text{ etc.} \end{aligned}$$

where $A^M{}_{/m}$ denotes covariant differentiation with respect to Christoffel brackets.

The affine paths of the connections (5.2) are, in fact, identical to the geodesics of a Riemann geometry with metric $ds = (g_{ij}dx^i dx^j)^{1/2}$, for the connections are projectively related to the Christoffel brackets. As for field equations, there are two fundamental affine tensors which may be obtained from (5.3). The concept of parallel displacement of a vector enables us to construct in the usual way a curvature tensor,

$$\bar{R}^i{}_{jkl} = R^i{}_{jkl} + \tfrac{1}{2}(\delta^i{}_j a_{k,l} - \delta^i{}_j a_{l,k})$$

where $R^i{}_{jkl}$ is the usual Riemann curvature tensor for metric g_{ij}. Another affine tensor, $S_{ij} = a_{i,j} - a_{j,i}$ may be constructed by the path dependence on parallel transfer of the length of a vector. From these two basic affine tensors, the usual field equations of general relativity may be constructed since $R_{ij} = \bar{R}_{ij} + S_{ij}$. Moreover, any further equations restricting S_{ij} will not affect the paths of the theory at all, so that a theory physically equivalent to general relativity is obtained.

Although (5.2) would seem to be the most natural restriction which reduces to $\underset{\vee}{\Gamma}{}^i{}_{jk} = 0$, when $a_l = 0$, the restriction $\underset{\vee}{\Gamma}{}^i{}_{jk} = 0$ itself also satisfies this condition. In this case we obtain from (5.1),

$$\Gamma^i{}_{jk} = \left\{ {i \atop j\,k} \right\} - \tfrac{1}{2}(\delta^i{}_j a_k + \delta^i{}_k a_j - g_{jk} a^i) \tag{5.4}$$

However, by the nature of the restriction, the symmetry between the differentiation of covariant and contravariant fields is destroyed. In fact we have

$$\Omega^i{}_{jk} = \left\{ {i \atop j\,k} \right\} - \tfrac{1}{2}(-\delta^i{}_j a_k + \delta^i{}_k a_j - g_{jk} a^i) \tag{5.5}$$

The symmetric connection (5.4) is that on which Weyl's theory is based (Weyl, 1952), and its appearance here can be seen as a direct consequence of the weakening of the product rule. For one effect of the new rule, implied by the connections (4.5), is that the length $A_i A^i$ of the vector A^i is not invariant on parallel transfer of A^i. We have

$$\frac{\delta(A_i A^i)}{\delta t} = A^i \frac{\delta A_i}{\delta t} + A_i \frac{\delta A^i}{\delta t} + a_k \frac{dx^k A_i A^i}{dt}$$

Taking $\delta A^i/\delta t = 0$, we have straight away from axiom 4 that $\delta A_i/\delta t = 0$, and so

$$\frac{\delta(A_i A^i)}{\delta t} = a_k \frac{dx^k}{dt} A_i A^i \tag{5.6}$$

Equation (5.6) was the basis of Weyl's development of the connection (5.4), but whereas it is here developed by dropping the product rule, in Weyl's theory axiom 4 is dropped (for instance, $\delta A^i/\delta t = 0$ does not imply $\delta A_i/\delta t = 0$ in Weyl's theory).

That nothing in addition to Weyl's theory is obtained from the connection (5.5) follows on construction of the curvature tensor from (5.5). This is found to differ from the Weyl curvature tensor by a gauge invariant term $\delta^i{}_j(a_{k,l} - a_{l,k})$, which is anyway one of the affine tensors of the theory. Finally, a third possible restriction, $\Omega^i{}_{\underset{\vee}{jk}} = 0$, also produces Weyl's theory. In this case, $\Gamma^i{}_{\underset{\vee}{jk}} = \frac{1}{2}(\delta^i_k a_j - \delta^i_j a_k)$, so that

$$\Gamma^i{}_{jk} = \begin{Bmatrix} i \\ j\,k \end{Bmatrix} - \tfrac{1}{2}(\delta^i_j a_k - \delta^i_k a_j + g_{jk} a^i)$$

$$\Omega^i{}_{jk} = \begin{Bmatrix} i \\ j\,k \end{Bmatrix} + \tfrac{1}{2}(\delta^i_j a_k + \delta^i_k a_j - g_{jk} a^i)$$

In this case, the roles of $\Gamma^i{}_{jk}$ and $\Omega^i{}_{jk}$ are reversed, $\Omega^i{}_{jk}$ giving the Weyl connection, so that Weyl's theory is produced again.

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